Information Flow

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Contents

Background

Fred Dretske: Knowledge and the Flow of Information (1981):

Definition 1. *A signal r* **carries the information** *that s is F iff the conditional probability of s's being F, given r (and the receiver's prior knowledge k), is 1 (but, given k alone, less than 1).*

Definition 2. *When there is a positive amount of information associated with s's being F: K* **knows that** *s is F iff K's belief that s is F is caused (or causally sustained) by the information that s is F.*

Barwise/Perry: Situations and Attitudes (1983):

Definition 3. *Let MO be a binary relation on some set MF of courses of events (partial functions from locations to situation types). MO is* **informational on a situation structure** *$\mathcal{M}$ if for each factual member e of $M \cap MF$ there is a factual $e' \in M$ such that $eMOe'$. The* **information** *of a meaningful course of events $e \in MF$ is the set $[\![e]\!]_{MO}$.*

Definition 4. *If every event of type E_1 is of type E_2, we have the following state of affairs:*

$$C := \text{at } l_u\text{: involves, } E_1, E_2\text{: yes.}$$

Definition 5. *The* **meaning** *of an indicative sentence ϕ is a relation $u[\![\phi]\!]e$ between utterance situations u and described situations e.*

$$DU := \text{at } \dot{l}\text{: speaking, } \dot{a}\text{; yes; addressing, } \dot{a}, \dot{b}\text{; yes; saying, } \dot{a}, \dot{\alpha}\text{; yes}$$

$d, c[\![$"sees that ϕ"$]\!]a, e \iff$ for every e' either

$d, c[\![$"ϕ"$]\!]e'$

or

in e: at $l = c($"sees"$)$: SO, a, e'; no

(e' is incompatible with what a sees or knows)

$d, c[\![$"doesn't see that ϕ"$]\!]a, e \iff$ there is an e' such that

not: $d, c[\![$"ϕ"$]\!]e'$

and

in e: at $l = c($"sees"$)$: SO, a, e'; yes

(e' is compatible with what a sees or knows)

Basic Definitions

Barwise/Seligman: Information Flow (1997):

Definition 6. *A* **classification** *is a triple* $A = \langle tok(A), typ(A), \models_A \rangle$ *of a set of tokens* $tok(A)$*, a set of types* $typ(A)$ *and a binary relation* $\models_A \subset tok(A) \times typ(A)$*.*

Definition 7. *An* **infomorphism** $f : A \rightleftarrows B$ *from* A *to* B *is a contravariant pair of functions* $f = (f^\wedge, f^\vee)$ *satisfying:*

$$f^\vee(b) \models_A \alpha \quad \Leftrightarrow \quad b \models_B f^\wedge(\alpha) \quad \forall b \in tok(B)\, \forall \alpha \in typ(A)$$

Definition 8. *An* **information channel** *is a family of infomorphisms* $\mathcal{C} = \{f_i : A_i \rightleftarrows C\}_{i \in I}$*.*

Information Flow

Let $tok(A)$ be the models of PA, $typ(A)$ formulae of $\mathcal{L}_1$, $tok(B)$ models of ZFC, with formulae of $\mathcal{L}_\epsilon$ and $\models_A$, $\models_B$ normal validity:

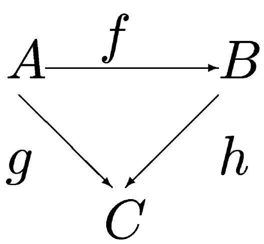
$$\mathcal{L}_1 \ni \alpha \mapsto \alpha^I \in \mathcal{L}_\epsilon$$

$$M_I \leftarrow M$$

$$M_I \models_{PA} \alpha \Leftrightarrow M \models_{ZFC} \alpha^I$$

Definition 9. *$a \models \alpha$ carries the information that $b \models \beta$ relative to C if:*

$$\exists c \in tok(C) \quad f(c) = a \wedge g(c) = b \wedge f(\alpha) \vdash_{Th(C)} g(\beta)$$



Regular Theories and Local Logics

Definition 10. *A* **theory** *is a pair* $T = \langle \Sigma, \vdash \rangle$ *of a set of types* Σ *together with a consequence relation* $\vdash \subset \mathcal{P}(\Sigma) \times \mathcal{P}(\Sigma)$*. A* **constraint** *of* T *is a sequent* $\langle \Gamma, \Delta \rangle \subset \vdash$*.*

Definition 11. *A sequent* $\langle \Gamma, \Delta \rangle$ *of sets of types is* **information about a token** a *or* **holds in** a *provided that if* a *is of every type of* Γ*, then* a *is of some type in* Δ*.*

Definition 12. *A binary relation* $\vdash$ *is* **regular** *if it satisfies the following:*
Identity: $\alpha \vdash \alpha$
Weakening: $\Gamma \vdash \Delta \quad \Longrightarrow \quad \Gamma, \Gamma' \vdash \Delta, \Delta'$
Global Cut:
$\Gamma, \Sigma_0 \vdash \Delta, \Sigma_1 \quad \forall \langle \Sigma_0, \Sigma_1 \rangle \in Part(\Sigma') \;\Rightarrow\; \Gamma \vdash \Delta$
T *is regular iff* $\vdash_T$ *is regular.*

Definition 13. *The* **theory generated by a classification** A *is* $Th(A) = \langle typ(A), \vdash_A \rangle$*, whose constraints are the set of sequents which hold in every token of* A*.*

Proposition 1. *Every regular theory is $Th(A)$ for some classification A.*

Definition 14. *A* **local logic** *is a quintuple $\mathcal{L} = \langle tok(\mathcal{L}), typ(\mathcal{L}), \models_{\mathcal{L}}, \vdash_{\mathcal{L}}, N_{\mathcal{L}} \rangle$ consisting of:*

- *a classification $cla(\mathcal{L}) = \langle tok(\mathcal{L}), typ(\mathcal{L}), \models_{\mathcal{L}} \rangle$*
- *a regular theory $th(\mathcal{L}) = \langle typ(\mathcal{L}), \vdash_{\mathcal{L}} \rangle$*
- *a subset $N_{\mathcal{L}} \subset tok(\mathcal{L})$ of* **normal tokens** *of $\mathcal{L}$ (tokens which satisfy all the constraints of $th(\mathcal{L})$)*

Definition 15. *$\mathcal{L}$ is* **sound** *if every token is normal. It is* **complete** *if every sequence satisfied by every normal token is a constraint of $Th(\mathcal{L})$, i.e. if every consistent sequence has a normal counterexample.*

Definition 16. *A* **logic infomorphism** *$f : \mathcal{L}_1 \rightleftarrows \mathcal{L}_2$ is a contravariant pair of functions such that:*

- *$f : cla(\mathcal{L}_1) \rightleftarrows cla(\mathcal{L}_2)$ is an infomorphism.*
- *$f^{\wedge} : th(\mathcal{L}_1) \rightarrow th(\mathcal{L}_2)$ is a theory interpretation.*
- *$f^{\vee}(N_{\mathcal{L}_2}) \subset N_{\mathcal{L}_1}$.*

Comparing Channels, Theories and Logics

Definition 17. *Let* $\mathcal{C}' = \{g_i : A_i \rightleftarrows C'\}_{i \in I}$ *and* $\mathcal{C} = \{f_i : A_i \rightleftarrows C\}_{i \in I}$ *be two channels.*

$$\mathcal{C}' \sqsubseteq \mathcal{C} \quad :\Longleftrightarrow \quad \exists r : C' \rightleftarrows C \ \wedge \ f_i = r \circ g_i \ \forall i$$

Definition 18. *Let* T_1 *and* T_2 *be two regular theories.*

$$T_1 \sqsubseteq T_2 \quad :\Longleftrightarrow \quad (\Gamma \vdash_{T_1} \Delta \Rightarrow \Gamma \vdash_{T_2} \Delta)$$

Definition 19. *Let* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *be two local logics.*

$$\mathcal{L}_1 \sqsubseteq \mathcal{L}_2 \quad :\Longleftrightarrow \quad th(\mathcal{L}_1) \sqsubseteq th(\mathcal{L}_2) \wedge N_{\mathcal{L}_2} \subset N_{\mathcal{L}_1}$$

Moving Theories and Logics

Definition 20. *The* **image** $f(T)$ *of a regular theory* $T = \langle \Sigma, \vdash_T \rangle$ *under* $f : \Sigma \to \Sigma'$ *is the theory with types* Σ' *and the consequence relation given by its consistent partitions:*

$$\langle \Gamma, \Delta \rangle \;\; f(T)\text{-cons.} \;\; :\Leftrightarrow \;\; \langle f^{-1}(\Gamma), f^{-1}(\Delta) \rangle \; T\text{-cons.}$$

Definition 21. *The* **image** $f(\mathcal{L})$ *of a local logic* $\mathcal{L}$ *on* A *under* $f : A \rightleftarrows B$ *is the local logic with classification* B*, regular theory* $f(th(\mathcal{L}))$ *and normal tokens* $f^{-1}(N_{\mathcal{L}})$.

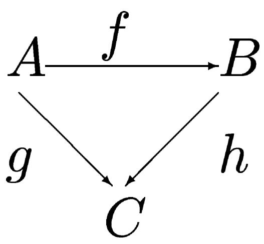
Proposition 2. *The image of a sound local logic is sound. The image of a complete local logic under a token surjective and type surjective infomorphism is complete.*

Definition 22. *The* **inverse image** $f^{-1}(T')$ *of a regular theory* $T' = \langle \Sigma', \vdash_{T'} \rangle$ *under* $f : \Sigma \to \Sigma'$ *is the theory with types* Σ *and the consequence relation given by:*

$$\Gamma \vdash_{f^{-1}(T')} \Delta \quad :\Longleftrightarrow \quad f(\Gamma) \vdash_{T'} f(\Delta)$$

Definition 23. *The* **inverse image** $f^{-1}(\mathcal{L})$ *of a local logic* $\mathcal{L}$ *on* B *under* $f : A \rightleftarrows B$ *is the local logic with classification* A*, regular theory* $f^{-1}(th(\mathcal{L}))$ *and normal tokens* $f(N_{\mathcal{L}})$*.*

Proposition 3. *The inverse image of a sound local logic under a token surjective infomorphism is sound. The inverse image of a complete local logic is complete.*



Definition 24. f-**Intro** *is the following inference rule:*

$$\frac{f^{-1}(\Gamma) \vdash_A f^{-1}(\Delta)}{\Gamma \vdash_B \Delta}$$

It preserves soundness, but not completeness.

Definition 25. f-**Elim** *is the following inference rule:*

$$\frac{f(\Gamma) \vdash_B f(\Delta)}{\Gamma \vdash_A \Delta}$$

It preserves completeness, but not soundness.

Definition 26. *The local logic* $Log_{\mathcal{C}}(B)$ *on* B **induced by the binary channel** $\mathcal{C} = \{g : A \rightleftarrows C, h : B \rightleftarrows C\}$ *is the logic* $Log_{\mathcal{C}}(B) = h^{-1}(g(Log(B))$. *It is, in general, neither sound nor complete.*

Proposition 4. *Let* g *be type surjective. A partition* $\langle \Gamma, \Delta \rangle$ *of* $typ(B)$ *is consistent in* $Log_{\mathcal{C}}(B)$ *iff* $\langle g^{-1}(h(\Gamma)), g^{-1}(h(\Delta)) \rangle$ *is the state description of some* $a \in tok(A)$. *A token* $b \in tok(B)$ *is normal in* $Log_{\mathcal{C}}(B)$ *iff it is connected to a token of* A.

Proposition 5. *Every local logic* $\mathcal{L}$ *on a classification* A *is of the form* $Log_{\mathcal{C}}(A)$ *for a binary channel* $\mathcal{C}$ *linking* A *to the classification* $Cla(SC(\mathcal{L}))$ *of the sound completion of* $\mathcal{L}$.

Future Work

- Study special cases.
- Find examples.
- Embed Kripke Structures as in Barwise (1997).
- Find out what all this has to do with information (flow).