What numbers might be

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In a famous passage of the *Grundlagen* (1884: §§60-68), Frege considers whether it is possible to introduce the concept of natural numbers by recognising the validity of the following inference:

(1) $$\frac{\text{There is a bijection between the } F\text{s and the } G\text{s}}{\text{the number of } F\text{s} \quad = \quad \text{the number of } G\text{s}}$$

Bob Hale and Crispin Wright have argued that Frege dismissed this project for the wrong reasons[1] and that (1) can indeed, if taken as a merely analytic and not, as Frege did, as a logical truth, provides a means of introducing the concept of a cardinal number (Hale & Wright 2001a: 4).

(1), Hale & Wright (2001a: 10) say, allows us to *infer* the existence of numbers from the truth of claims that some concepts can be one-to-one correlated.[2] This is so, they think, because its validity is underwritten by a general principle stating identity conditions for numbers called 'Hume's principle':

(HP) For any concepts F and G, there is a bijection between the F and the G iff the number of Fs = the number of Gs.

In a more formal notation, **HP** shows how identity conditions for terms constructed by the term-forming operator $\mathsf{N}(\dots)$ may be given in a second-order language:

(HP') $$\forall F, G\ \exists R\ \forall x\ ((Fx \to \exists y(Gx \wedge \forall z(R(x,z) \to z = y)) \wedge Gx \to \exists y(Fy \wedge \forall z(R(z,x) \to z = y))) \longleftrightarrow \mathsf{N}x(Fx) = \mathsf{N}x(Gx)$$

Hale & Wright (2001a: 4) define the "neo-fregean programme" as consisting in the conjunction of the logical claim that all the fundamental laws of arithmetic are derivable from a consistent system which consists of second-order logic together with Hume's Principle[3] and a philosophical claim to the effect that Hume's Principle may be "viewed as an *implicit* definition, effecting an introduction of a sortal concept of cardinal number and, accordingly, as being analytic of that concept" (Hale & Wright 2001a: 4).

They take the logical claim to be substantiated by "Frege's Theorem", i.e. the result that Hume's Principle, added to a suitable system of second-order logic, suffices for proof of the Dedekind-Peano axioms[4] and by the result that Frege arithmetic (second-order logic plus a restricted form of naïve second order comprehension which is closely related to Hume's Principle) is consistent if second-order arithmetic is.[5]

Here is an immediate worry concerning the ontological consequences of **HP**: how can it be that two statements are a priori equivalent if they differ in ontological commitment, the left-hand side of the equivalence

[1] Frege rejects it, in (Frege 1884: §66), for what has become known as the Caesar problem.

[2] Cf.: "The effect of Hume's principle is to fix the truth-conditions of identity statements featuring canonical terms for numbers as those of corresponding statements asserting the existence of one-one correlations among the appropriate concepts. Given statements of the latter sort as premisses, the truth of such identities (and hence the existence of numbers) may be inferred." (Hale & Wright 2001a: 10)

[3] In view of Gödel's incompleteness theorem, this had better not be extended to all *truths* of arithmetic (Kneale & Kneale 1962: 723-724). (Wright 1983: 131) interprets Gödel as showing that logical truth defies complete deductive characterization.

[4] Due to Parsons (1965), also cf. Smiley (1981: 54), Heck (1993), Boolos & Heck (1998); a proof is sketched in (Wright 1983: 158-169) and in the appendix to (Boolos 1990).

[5] This has been established, independently, both by Burgess (1984) and Boolos (1987). A detailed proof is in the appendix to Boolos & Heck (1998).

featuring singular terms for numbers while the right-hand side does not? Hale and Wright, after all, do not take the 'number of' operator N to be a logical constant: even if it is a term-forming operator which applied to concepts gives singular terms having a sense, this will not give them a guarantee that they also possess a reference (cf. Rumfitt 2003: 208). The underlying logic, therefore, should split **HP** into two principles:[6]

HP⁻ For any concepts F and G, if there is such an object as the number of Fs or the number of Gs, there is bijection between the F and the G iff the number of Fs $=$ the number of Gs

C For any concept F, there is such an object as the number of Fs

Splitting **HP** into **HP⁻** and **C** makes it clear that some further restriction is needed. **C**, if left unrestricted and applied to the concept *being self-identical*, immediately entails that there is a number of all objects (Boolos 1990: 274). **HP**, if it thus entails the existence of anti-zero, a number greater than any other number, is "incompatible with Zermelo-Fraenkel set theory plus standard definitions, on the usual and natural readings of the non-logical expressions of both theories" (Boolos 1997: 260). A Fregean, as Rumfitt (2001b: 517) has argued on his behalf, is best advised to deny the presupposition that cardinals are ordinals, on account of the fact that they give answers to "how many?", not "how much?" questions.[7]

This move ties the existence of natural numbers to "how many?" questions: there is such a thing as the numbers of Fs, we may say, iff the question "How many Fs are there?" has a uniquely correct answer (Rumfitt 2001b: 524). But how are we to answer such a question? Frege tells us in the *Grundgesetze* (cf. Rumfitt 2001b: 525–526):

> Wenn wir die zu einem Begriffe $\Phi(\zeta)$ gehörende Anzahl bestimmen, oder, wie man gewöhnlich sagt, wenn wir die unter den Begriff $\Phi(\zeta)$ fallenden Gegenstände zählen, so ordnen wir diese den Zahlwörtern von Eins an der Reihe nach zu bis zu einem Zahlworte "N", das dadurch bestimmt wird, dass die zuordnende Beziehung den Begriff $\Phi(\zeta)$ in den Begriff "Glied der Reihe der Zahlwörter von "Eins" bis "N" " und dass die umgekehrte Beziehung diesen Begriff in jenen abbildet. Dann bezeichnet "N" die gesuchte Anzahl; d.h. N ist diese Anzahl. (Frege 1893: §108)[8]

Frege's idea, in other words, is that we count objects by correlating them one-to-one with an initial segment of a sequence which is, in the terminology of Rumfitt (2001b: 527), "a simple tally", i.e. which has a beginning, is functional and has an irreflexive strong ancestral. Contrary to Rumfitt, however, I do not think that Frege's choice of number-words is "obviously inessential". It is true that number-words will run out after $\aleph_0$ and that we need a more general notion of equinumerosity with a bounded segment of a strictly well-ordered sequence after that. But Frege's idea that counting is establishing a 1-1 correspondance with number words is still significant, for it was what made him focus on **HP** in the first place.[9]

More importantly, 1-1 correlation of numbers with number-words may provide us with the materials to establish the antecedent of **HP⁻**. Frege's main reason, as is well known, for regarding numerals as singular terms (and, correspondingly in his system, numbers as objects) was that they give complete answers to "how many?" questions. These answers, however, may readily be expanded into sentences in which the numerical expressions occur as adjectives. Hence the attraction of what Dummett (1991: 99) calls the "radical adjectival strategy", whereby "equations and other arithmetical statements in which numerals apparently figure as singular terms are to be explained [...] by *transforming* them into sentences in which number-words occur only adjectivally". The answer to "how many philosophers does it take to fix a light-bulb", then, is not "two", but "two philosophers are needed to fix a light-bulb", in the logical form of which we find a numerical

[6] Cf. Rumfitt (2001b: 522), Rumfitt (2003: 209).

[7] This was Frege's reason in the *Grundgesetze* not to identify natural numbers with real numbers (Frege 1903: §157) and he says in the *Grundlagen* that cardinal numbers, in contrast to "Cantor's numbers", contain no reference to any fixed number (Frege 1884: §85). It will later be argued that this latter statement should be modified. CHECK Grundlagen §10: "The numbers are related to one another quite differently from the way in which the individual specimens of, as it might be, a species of animal are, for it is in their nature to be arranged in a definite order of precedence".

[8] "If we determine the number belonging to a concept word $\Phi(\zeta)$ – or, as one ordinarily says, if we count the objects falling under the concept $\Phi(\zeta)$ – then we successively co-ordinate these objects with the number-words from "one" up to a number-word "N". This number-word is determined through the co-ordinating relation's mapping the concept $\Phi(\zeta)$ into the concept "member of the series of number-words from "one" to "N" " and the converse relation's mapping the latter concept into the former. "N" then designates the number sought; i.e. N is this number."

[9] Is is true that he regarded, in his review of Husserl (1891), the correlation with numerals as a "detour" (Frege 1894: 319).

quantifier:[10]

(2) $\exists x, y$ (x is needed to fix a lightbulb $\wedge$ y is needed to fix a lightbulb $\wedge$ $x \neq y$ $\wedge$ $\forall z$(z is needed to fix a lightbulb $\rightarrow$ $(z = x \vee z = y))))$

This brings us rather close to a position defended by (Hodes 1984) and (Hodes 1990): numerals are singular terms but not all singular terms have the semantic function of effecting reference to an object. Numerals have the semantic function of "encoding" exact cardinality quantifiers.

As Rumfitt (2001a: 50) argues, these numerical quantifiers have to be understood as indicating a position in a series of quantifiers ordered by implication. In order to understand "There are at least three men", on this account, one has to appreciate that it belongs in a sequence whose immediate predecessor is "Some men x and y are distinct" and "Some men x, y, z and w are such that x, y, z and w are distinct" (Rumfitt 2001a: 53).

If counting consists in a 1-1 correlation between numbers and numerical quantifiers, we are in a position to motivate a restriction of **C** to some subclass of sortals. We may impose, e.g., a condition of countability, or disqualify concepts which are indefinitely extensible or impose some other restriction.[11]

The recourse to numerical quantifiers also solves a problem with respect to the number 0. Frege's account of natural numbers as *Anzahlen*, presupposes, as Rumfitt (2001b: 527) has noted, that determining the number belonging to some concept F is the counting the Fs. This is not true for the case of zero: when we determine that the number belonging to the concept *being a predecessor of Romulus on the throne of Rome* is zero, we are not counting anything (Frege 1894: 328). There is, however, a numerical quantifier.

The most important advantage of numerical quantifiers, however, is that they give us an answer to the question what numbers are and thus a way of establishing the antecedent of $\mathbf{HP}^-$. We have the following correlations:

$$\begin{array}{rclll} 0 & \rightsquigarrow & \exists_0 x Fx & := \forall x \neg Fx \\ 1 & \rightsquigarrow & \exists_1 x Fx & := \neg\forall x \neg Fx \wedge \forall x \forall y((Fx \wedge Fy) \rightarrow x = y) \\ n+1 & \rightsquigarrow & \exists_{n+1} x Fx & := \exists x(Fx \wedge \exists_n y(Fy \wedge x \neq y)) \end{array}$$

For each numerical quantifier in the series, we need one more variable not already occurring in its predecessor. For the n-th numerical quantifier, then, we'll need $n+1$ variables. Their semantic role is to coordinate argument places, to tell us in what places our assignment of values have to be coordinated.

In recent work, Kit Fine (2003) has studied this coordination in more detail. In his defense of semantic relationalism, the thesis that there are external semantic relations, i.e. relations not supervenient on intrinsic semantic features of their relata, he emphasised the fact that the simultaneous assignment of values to different variables must provide them with a coordination scheme, i.e. tell which occurrences are to be coordinated with which other occurrences of the same or different variable.

In view of his earlier work on neutral relations, Fine (2000) argued that for some relations, the coordination scheme has to be distinguished from the relation itself. *Greater than*, the order of the number series, may provide an example of such a neutral relation: we do not want to say that, whenever some number n is greater than some other number m, we have two relational complexes, n and m together with the relation *greater than* and n and m together with the relation *smaller than*. Instead, we have to distinguish the relation from the order it imposes on its relata.

Fine discusses two ways to achieve this. Positionalism is a position that reifies argument places, and explicitly correlates them with the relata of the relation. Exemplification of the relation must then be understood to be relative to an assignment of objects to argument-places (Fine 2000: 11). The main problem of this view is that it is not clear what argument places might be.The other drawback is that positionalism is unable to account for strictly symmetric relations, i.e. relations R such that a's being R-related to b is the *same*

[10] Frege says, in §57 of the *Grundlagen* that the proposition "Jupiter has four moons" can be 'converted' (*umsetzen*) into "The number of Jupiter's moons is [identical with the number] four", but there are many reasons to be skeptical of this kind of transformation (cf. Rumfitt 2001a: 49).

[11] It was because he took "object" not to be "a proper concept-word" (*eigentliches Begriffswort*) that Wittgenstein (1922: §4.1272) held that "non-sensical to speak of the *total number of objects*".

relational fact than b's being R-related to a (Fine 2000: 17). This, however, only holds if we do not identify argument-places with other entities.

Anti-positionalism, on the contrary

If $a_1, \ldots, a_n$ are the distinct (nonoverlapping) constituents of s and t is the result of substituting b_1 for a_1, ...and b_n for a_n, then each b_i in t occupies the same position in t as a_i occupies in s. Position is preserved under substitution.

If we are anti-positionalists about relations, we can define argument places as equivalence classes

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