Paradox Philosophy

Course at Akita International University
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Day 1: February 10, 2020

Course description. Why is it that some things are too big to exist? Why can't things not include themselves or be too big to be thought about? Does Achilles ever overtake the tortoise, is motion even possible? Does the liar who says "I'm lying" even say something; if so, is it true what he says? If everyone does what is best for him no matter what the others do – will then all be better off? if I were offered a blessed life in the Matrix, filled with only the experiences I cherish most, should I take the offer? is there anything valuable at all?
These are interesting, but also difficult question. In this beginners' course, students will become acquainted with some ways of asking them and be encouraged to think about answers.
In this course, we will encounter five different paradoxes, hitherto unsolved, from different areas of philosophy. Attending students will get brief presentations beforehand, will learn about the background assumptions and the theoretical context, try their hands at solutions (group work of about 30 minutes) and then be presented with an overview of extant diagnoses, being given the opportunity to critically discuss them.

Target audience. The course is designed for students who are ambitious enough to test their critical thinking skills at some of the most difficult problems at all. They will benefit first, by learning humility, that nothing is quite as easy as it first appears, and second, by appreciating their own intellectual progress as they revise, change, refine and develop a first response to the problems, which are likely to accompany them for the rest of their lives. An important task is to justify and defend their positions; students will thus also improve their debating skills.

Schedule. We will devote each day to a different family of paradoxes:

- Monday: Paradoxes of size – the set-theoretic paradoxes (Cantor, Russell)
- Tuesday: No class.
- Wednesday: Paradoxes of motion – Zeno, but also more contemporary versions (0-valued quantities, instantaneous velocity, in-/finite time)
- Thursday: Paradoxes of truth – the Liar, semantic paradoxes (Grelling, Curry)
- Friday: Paradoxes of decision-theory and knowledge – Prisoners' Dilemma, Newcomb, the unexpected examination
- Monday: Paradoxes of value - the experience machine, the toxin puzzle, bootstrapping

Objectives. Critical thinking – understand a problem, devise new problem-solving strategies, defend these against critical objections.

Office hours. Suggestions welcome.

Group work. Have a look at this selection of paradoxes from Clark (2007).

What is a paradox?

Paradoxes have a long history of both provoking and frustrating thoughs, often challenging unproved assumptions and uncovering surprising equivocations. Sainsbury (1995, 1) characterises them as follows:

> This is what I understand by a paradox: an apparently unacceptable conclusion derived by apparently acceptable reasoning from apparently acceptable premises.

Similarly, Quine (1962, 1) says:

> …a paradox is just any conclusion that at first sounds absurd but has an argument to sustain it.

But is a paradox just a puzzle? I would think a good, interesting paradox is more. Some 'paradoxes' (scare quotes) are just fallacies, some are real dilemmas, some perhaps even antinomies.

Things and their groupings

Things group together in various ways: many different little parts compose a whole, different notes a tune, different people a family. Some of the properties of the groupers are inherited by the group, some not: if the little parts are out of wood, the whole is, at least partly, wooden, but the note may be harmonious while the tune is not, the family may be five while each member of it is just one.

Sometimes, groups have the same properties as their members in the same way, sometimes they have the same properties, but in different ways. The group of large things is itself large, but in another way; the group of all the things is something I like, but in another way. Not so with other properties: the group of things in Japan is in Japan, the group of precious things is itself precious, the group of all groups that were mentioned here today is itself a group.

Groups do not just have properties, and sometimes the same properties than their members, they can also be defined by them. I can characterise the very same group in two different ways, called 'extensionally', by enumeration, or 'intensionally', by a condition all and only its members fulfill:

$$X := \{\text{Huey}, \text{Dewey}, \text{Louie}\} \qquad X := \{x \mid x \text{ is a child of Donald Duck}\}$$

For many groups, I need to impose further conditions: for some things to be a *family*, for example, it is not enough that they exist; they also have to be (in some legal sense) relatives of each other. For other things, more 'material' relations are needed: for a number of wooden planks to constitute a ship, for example, they have to be in contact with each other and together be arranged in a certain form.

For the 'lightest' groups, what mathematicians call "sets", no such conditions are required: the mere existence of the members guarantees the existence of the set – or so it seems at least!

In the case of sets, I have defined the very same set of exactly three members above, extensionally and by enumeration on the left, intensionally and by a condition on the right.

Though this is not required in all cases,[1] several phenomena suggest treating 'groups' as new entities, something 'over and above' their members:

1. For many – some say: for all – 'ordinary' uses, talk about groups can be 'paraphrased away', translated into talk not about groups but just plurally about several things. "Donald Duck takes the group of his children to the movies", e.g., is just a funny way of saying the same thing as does "Donald Duck takes his children (or: Huey, Dewey and Louie) to the movies".

- While the members are many, the group is one.
- In many cases, the group is somewhat more abstract, more aloof, more 'formal' than its members.
- In many cases, the group seems to have different properties than its members.

Once we make this step for sets, however, paradox looms.

Cantor and modern set theory

The German mathematician Georg Cantor was interested in the use of infinities in mathematics. The infinity of the natural numbers puzzled already the Greeks:

$$\mathbb{N} := \{1, 2, 3, 4, 5, 6, \dots\} \qquad \mathbb{N} := \{x \mid x \text{ is a natural number}\}$$

How 'big', in an intuitive sense, is this group? Is it bigger, for example, than the set of odd natural numbers:

$$\mathbb{E} := \{2, 4, 6, 8, 10, 12, \dots\} \qquad \mathbb{E} := \{x \mid x \text{ is a natural number} \wedge x \, x \text{ is even}\}$$

At least to me, a long time ago, the answer seemed simple: *obviously*, $\mathbb{N}$ is bigger than $\mathbb{E}$!

- First, all members of $\mathbb{E}$ are members of $\mathbb{N}$, but not all members of $\mathbb{N}$ are members of $\mathbb{E}$: for example, 1, 3, 5 are not!
- Second, the set $\mathbb{E}$ is defined by an extra clause ("being odd") which is not true of all the members of $\mathbb{N}$: to its a real condition and it is thus 'harder' to be a member of $\mathbb{E}$ than it is to be a member of $\mathbb{N}$ ("it's not enough to be a natural number, you also have to be divisible by 2…").

It was therefore surprising that Cantor defended the opposite answer: in the mathematically relevant sense, both sets are equally big – they have the same *cardinality*. They have the same cardinality because there is a one-to-one correspondence (mathematically: a bijective function) between their members: it's multiplication by 2!

- for every member of $\mathbb{N}$ there is exactly one member of $\mathbb{E}$, which is the double of that first member of $\mathbb{N}$;
- for every member of $\mathbb{E}$ there is exactly one member of $\mathbb{N}$, which is the half of that first member of $\mathbb{E}$.

Not only does the set of natural numbers have the same cardinality than the set of even numbers, it even has the same cardinality as the set $\mathbb{Q}$ of rational numbers! But not, Cantor showed, as the set $\mathbb{R}$ of real numbers.

As the usefulness of sets became apparent and set-theory became the foundation of present-day mathematics, a first paradox ensued: what about all sets – do they form a set? Intuitively, if we know what a set is (there's not much to it, it's just a bare collection), we can define the set of all sets as follows:

$$\mathbf{V} := \{\mathbb{N}, \mathbb{R}, \mathbb{E}, \text{the set of all cows}, \{\text{Huey}, \text{Dewey}, \text{Louie}\} \dots\}$$

or, by a condition, as follows:

$$\mathbf{V} := \{x \mid x \text{ is a set}\}$$

If we do so, however, we can ask: is V a set?

- If it is, then it is a member of V (because that's how V was defined): but then $V \in V$, i.e. the set contains itself – but does it then also contain other sets, such as the set of all cows? what is left of it, if we strip it of just one of its (many) members, namely of V?
- But if it is not, then not everything that is definable as above, not every collection of things is a set – what further conditions are required?

Worse was to come, however: the so-called 'diagonalisation argument' Cantor gave to show that the cardinality of $\mathbb{R}$ was strictly greater than that of $\mathbb{N}$ (mathematically: that $\mathbb{N}$ can be embedded into $\mathbb{R}$ but that there is no one-to-one correspondance of $\mathbb{R}$ to $\mathbb{N}$) can be generalised to Cantor's theorem: that there are always more subsets than members (Cantor, 1892). If we start with a set X and form the set of its subsets (the so-called "powerset of X", $\mathcal{P}(X)$), we get a set of higher cardinality than X. If there were a set V of all sets, then we would have $|\mathcal{P}(V)| \leq |V|$, i.e. the cardinality of its powerset would not be greater than that of the set V itself (for all the subsets of V are also sets), which contradicts Cantor's theorem.

Frege's definition of number and Russell's paradox

Despite its unapplicability to V, the concept of cardinality proved very fruitful. The mathematician and philosopher Gottlob Frege, arguably the grandfather of modern Western philosophy, used it to define the concept of natural number (Frege, 1884). In the same way in which directions are what parallel lines have in common, Frege said, numbers are what equinumerous sets (ie. sets of the same cardinality) have in common. By identifying sets, in effect, with conditions, Frege wanted to provide a purely logical foundation for the whole of mathematics.

Unfortunately, this project foundered. It was in the first volume of Frege's major work, "The Fundamental Laws of Arithmetic" (Frege, 1893), that the English philosopher Bertrand Russell discovered a contradiction. Russell showed that Frege's rules permitted the definition of a set, the existence of which allowed the derivation of a contradiction from the system and thus showed the system to be inconsistent (i.e.: useless).

The set in question was the set of all sets that do not contain themselves. Some sets do not contain themselves, for example the set of cows. It's a set, and not a cow, and to be a member of it something has to be a cow, so it is not a member of itself. There are many other such sets: the set of people, the set of colours, the set of Donald Duck's childrens and so on. So let us define the set of them and call it "Hugo":

$$\mathbf{H} \;:=\; \{x \mid x \notin x\}$$

Now we know what Hugo is and we know that Hugo is a set, we can ask a simple question: does it, or does it not, contain itself? The problem is that neither answer is possible:

- yes, that is: $H \in H$. But then Hugo has to satisfy the condition that defines it, which is: *not containing itself*; so: if Hugo contains itself, it does not contain itself;
- no, that is: $H \notin H$. But then Hugo *does* satisfy the condition that defines it, and so it *does* contain itself; so: if Hugo does not contain itself, it does not contain itself.

This means that we are in a very uncomfortable situation. The problem is not that one possible answer is 'unstable' and leads to its negation. This is perfectly acceptable (it's called a proof 'by reduction to absurdity'): if the answer 'yes' to a yes/no question leads to the answer 'no', then we know that the answer is 'no'.

The problem is rather that *both* answers to the question are 'unstable' in this way; and that because the question is a yes/no-question, no 'middle ground' is possible. We have obtained a

proof of the following claim:

$$\text{Hugo} \in \text{Hugo} \iff \text{Hugo} \notin \text{Hugo}$$

This is a straight contradiction, a reduction to absurdity of some of the assumptions that have gone into its derivation. The only such assumption was a completely innocuous one – that there is a set of all the sets that do not contain themselves.

We *have to* conclude that there is no such set. But it is difficult to see why not. Frege was devasted:

> Your discovery of the contradiction caused me the greatest surprise and, I would almost say, consternation, since it has shaken the basis on which I intended to build arithmetic, [...] In any case your discovery is very remarkable and will perhaps result in a great advance in logic, unwelcome as it may seem at first glance. (Frege, letter to Russell, 22 June 1902, quoted in van Heijenoort (1967, 127–128))

Of Frege's reaction, Russell said in a letter published for the first time in 1967:

> As I think about acts of integrity and grace, I realise that there is nothing in my knowledge to compare with Frege's dedication to truth. His entire life's work was on the verge of completion, much of his work had been ignored to the benefit of men infinitely less capable, his second volume was about to be published, and upon finding that his fundamental assumption was in error, he responded with intellectual pleasure clearly submerging any feelings of personal disappointment. (Russell to van Heijenoort, 23 November 1962, in van Heijenoort (1967, 127))

In most modern set-theories, not every condition defines a set. Sets are constructed axiomatically, as things that satisfy certain axioms. But we still do not know what they are.

References

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